On the legacy of Yvonne's foundational Acta paper

Sergiu Klainerman

Princeton University

January 25, 2024

PLAN OF TALK

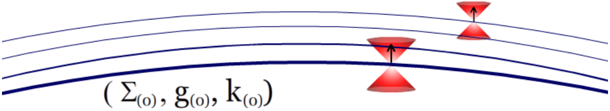
1. YCB 1952 Acta Paper. *Existence theorem for the Einsteinian gravitational field equations in the non-analytic case*.
2. Existence results based on the energy method. Bounded L^2 theorem.
3. Kirchoff-Sobolev and the breakdown criterion.
4. Vectorfield method, null condition, stability of Minkowski.
5. Kerr stability.

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = T_{\alpha\beta}$$

- EINSTEIN VACUUM (EV). $\qquad R_{\alpha\beta} = 0.$
- ISOTHERMAL COORDINATES $\qquad \Box_g x^\alpha = 0,$
  DeDonder(1921), Lanczos(1922).

$$\mathbf{g}^{\mu\nu} \partial_\mu \partial_\nu \mathbf{g}_{\alpha\beta} = N_{\alpha\beta}(\mathbf{g}, \partial \mathbf{g}).$$

- INITIAL DATA SETS. $(\Sigma_{(0)}, g_{(0)}, k_{(0)}) \ + $ Constraints
- YCB, YCB-GEROCH $\quad$ Initial data set $\longrightarrow$ unique MFGHD.



Yvonne Fourès Bruhat(1950-1952)

- C.R.A.S.(Feb.1950), -J. Hadamard.

  Théoréme d'existence .... dans le cas non analytique

- Acta Math(1952) - *Théoréme d'existence pour certains systémes d'equations aux dérivées partielles non-linéaires.*

- Local exist and unique for 2^{nd} order hyperbolic systems.
- Kirchoff-(Sobolev1936) formula.
- Application to EV using wave coordinates.

$$g_{(0)} \in C^5(\Sigma_0),\ k_{(0)} \in C^4(\Sigma_0).$$

KIRCHOFF-SOBOLEV (KS) PARAMETRICES

KS FORMULA.

$$\square_{\mathbf{g}}\phi = f, \qquad \phi\big|_{\Sigma_0} = \partial_t\phi\big|_{\Sigma_0} = 0.$$

$$4\pi\phi(p) = \int_{\mathcal{N}^-(p)} wf - \int_{\mathcal{N}^-(p)} \mathrm{Err}(\nabla^2 \operatorname{tr}\chi, \ldots)\,\phi$$

MAIN CALCULATION.

$$\square\big(w\delta(u)\big) = 4\pi\delta(p) - \mathrm{Err}(\nabla^2 \operatorname{tr}\chi, \ldots)\,\delta(u).$$

- Eikonal eq. $\quad \mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0, \quad u|_{\mathcal{N}^-(p)} = 0.$
- Expansion. $\quad \operatorname{tr}\chi = \square u.$
- Transport eq. $\quad -2\frac{d}{ds}w - w\operatorname{tr}\chi = 0, \quad sw(p) = 1.$

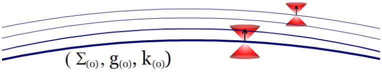
Hadamard(1932). Infinitely many transport eqts.

Sobolev(1936). Scalar quasilinear wave eqts.

YCB(1952). Systems of quasilinear wave eqts.

BREAKDOWN CRITERION

- Eardley-Moncrief(1982) *" Global Existence for YM in $\mathbb{R}^{1+3}$".*
- K-Rodnianski(2010). *" On a breakdown criterion in GR".*
- Qian Wang(2012). *"Improved breakdown criterion ...".*



$$\mathbf{g} = -n^2 dt^2 + g_{ij} dx^i dx^j$$

THEOREM. Assume Σ_t maximal (or CMC). Then spacetime can be continued past $t = t_*$ as long as

$$(*) \qquad \int_0^{t_*} \left(\|k(t)\|_{L^\infty} + \|\nabla \log n(t)\|_{L^\infty} \right) < \infty.$$

EULER EQUATION. Beale-Kato-Majda

MAIN STEPS

1. Curvature L^2-bounds, energy estimates.

$$(*) \quad \Rightarrow \quad \|\mathrm{Rm}(t)\|_{L^2} < \infty$$

2. Higher derivatives L^2-bounds, energy estimates.

$$(*) \ \& \ \|\mathrm{Rm}(t)\|_{L^\infty} < \infty \ \Rightarrow \quad \|\mathbf{D}^s \mathrm{Rm}(t)\|_{L^2} < \infty$$

3. Curvature L^∞-bound

$$(*) \ \& \ \|\mathrm{Rm}(t)\|_{L^2} < \infty \ \Rightarrow \quad \|\mathrm{Rm}(t)\|_{L^\infty} < \infty$$

- Step 3. *KS applied to* $\quad \Box_{\mathbf{g}} \mathrm{Rm} = \mathrm{Rm} * \mathrm{Rm}.$

$$\mathrm{Rm}(p) \quad = \quad \int_{\mathcal{N}^-(p)} w \, \mathrm{Rm} * \mathrm{Rm} - \int_{\mathcal{N}^-(p)} \mathrm{Err}(\nabla^2 \mathrm{tr}\, \chi \ldots) \cdot \mathrm{Rm}.$$

- Curvature flux along $\mathcal{N}^-(p)$ controls $\mathrm{Inj}\big(\mathcal{N}^-(p)\big)$ + bounds for $\nabla^2 \mathrm{tr}\, \chi$...Series of 4 papers by K-Rodniansk.

RESULTS BASED ON ENERGY ESTIMATES

$$\Box\phi = f; \quad \phi(0) = \partial_t\phi(0) = 0.$$
$$\|\partial\phi(t)\|_{L^2(\mathbb{R}^n)} \le \int_0^t \|f(s)\|_{L^2(\mathbb{R}^n)}$$

Gains a derivative by comparison to the Kirchoff formula.

PRE ACTA PAPERS.

- Friedrichs-Lewy(1928), *On the uniqueness and domain of dependence of the solutions to the initial value problem for linear hyperbolic differential equations*.
- Schauder(1935). *The initial value problem for a quasilinear hyperbolic equation of second order in any number of independent variables"*
- Sobolev(1936).

RESULTS BASED ON ENERGY ESTIMATES

POST ACTA RESULTS

- Leray(1952). *"Lectures on Hyperbolic Equations"* (IAS).
- Friedrichs(1954). Symmetric hyperbolic systems.
- Fischer-Marsden(1972). Energy +Sobolev+interpolation.

$$g_{(0)} \in H^s(\Sigma_0), \; k_{(0)} \in H^{s-1}(\Sigma_0), \qquad s > \frac{3}{2} + 1.$$

- K- R(2005). *" Rough solutions of the Einstein-vacuum equations".* $s > 2$.
- K- R-Szeftel(2015). *"Bded L^2-curvature theorem".* $s = 2$.
- Critical exponent $s = \frac{3}{2}$.

QUESTION: Is there a scale invariant version of well-posedness ?

Global well-posedness= Stability of Minkowski

STABILITY OF MINKOWSKI SPACE

YCB(1973) *Une theoreme d'instabilité por certains equations hyperbolique nonlineaires:* Einstein conjecture?

Linearized EV in the wave gauge $\Rightarrow$ logarithmic divergence.

- John(1976), K(1980). Use uniform decay to derive long time existence results for quasilinear wave equations.
- Morawetz(1961), K(1985)). Vectorfield method.
- Christodoulou(1986), K(1982, 1986). Null condition.
- Christodoulou-K(1993). *Stability of Minkowski space*.
  - Flexible gauge choice.
  - Brings back the use of characteristics.
  - Adapted vectorfield method to derive decay.
  - Intrinsic version of the null condition
- Lindblad-Rodnianski(2005). *Global existence in the Einstein Vacuum equations in wave co-ordinates.* Weak null condition.

OTHER GLOBAL STABILITY RESULTS

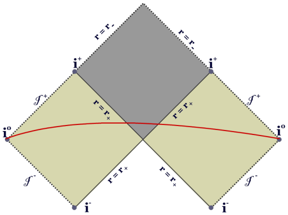
- Vacuum. K-Nicolo(2003), Bieri(2008), Huneau(2018), Shen(2023).
- Matter. Zipser(2009), LeFloch-Ma(2016), Qian-Wang(2020), Fajman-Joudioux-Smulevici(2020), Lindblad-Taylor(2020), Ionescu-Pausader(2022), Huneau-Stingo-Wyatt(2023).

OPEN QUESTION. Is there a translation invariant (without weights) version of the stability of Minkowski space ?

STABILITY OF KERR

CONJECTURE[Stability of (external) Kerr].

Einstein vacuum, asymptotically flat, perturbations of a given exterior Kerr ($\mathcal{K}(a, m)$, $|a| < m$) initial conditions have max. future developments converging to **another** Kerr solution $\mathcal{K}(a_f, m_f)$.



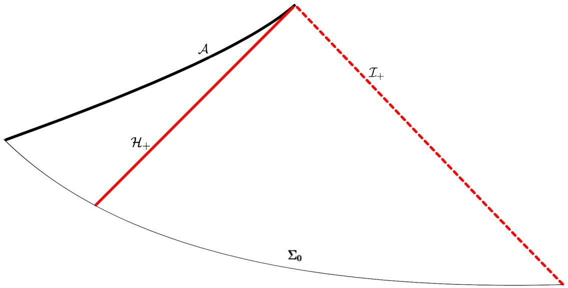
THEOREM "True" if $|a|/m \ll 1$.

- MAIN[K-Szeftel(2021)]
- GCM PAPERS[K-Szeftel(2019), Shen(2022)]
- WAVE PAPER[Giorgi-K-Szeftel(2022)]

STABILITY OF SLOWLY ROTATING KERR

THEOREM[KI-Szeftel(2021)] *The FMGHD of a general, (AF) IDS, close to the IDS of a* $\mathcal{K}(a_0, m_0)$, $|a_0|/m_0 \ll 1$

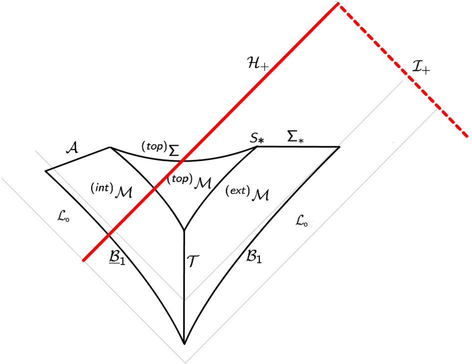
- *Has a complete future null infinity* $\mathcal{I}^+$
- *Converges in* $\mathcal{J}^{-1}(\mathcal{I}^+)$ *to a nearby* $\mathcal{K}(a_\infty, m_\infty)$ *with parameters* (a_∞, m_∞) *close to* (a_0, m_0).



Spacetime is constructed by continuity as a limit of finite GCM admissible spacetimes.

GCM ADMISSIBLE $\mathcal{M} = {}^{(int)}\mathcal{M} \cup {}^{(ext)}\mathcal{M} \cup {}^{(top)}\mathcal{M}$

- S_* - **GCM** surface.
- (a, m), "**axis**".
- Σ_*- **GCM** hypers.
- $({}^{(ext)}\mathcal{M}, u)$
- $({}^{(int)}\mathcal{M}, \underline{u})$
- **PG**-structures
- **Bootstrap.**



$(\mathcal{M}, a, m, \textbf{axis})$ are continuously **upgraded**.

REMARK. Gauge is initialized from the **future** with no reference to the initial data! Modifies the initial layer foliation!-Recoil!